

Gauge transformation and symmetries of the commutative multi-component BKP hierarchy

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Abstract

In this paper, we defined a new multi-component BKP hierarchy which takes values in a commutative subalgebra of $gl(N, \mathbb{C})$. After this, we give the gauge transformation of this commutative multi-component BKP (CMBKP) hierarchy. Meanwhile we construct a new constrained CMBKP hierarchy which contains some new integrable systems including coupled KdV equations under a certain reduction. After this, the quantum torus symmetry and quantum torus constraint on the tau function of the commutative multi-component BKP hierarchy will be constructed.

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1 Introduction

The KP and Toda lattice hierarchies as completely integrable systems have many important applications in mathematics and physics including the theory of Lie algebras'

representation, orthogonal polynomials and random matrix model [1–6]. The KP hierarchy has many kinds of reduction or extension, for example the BKP, CKP hierarchies and so on. As important sub-hierarchies of the KP hierarchy, the constrained KP (cKP) hierarchy, the constrained BKP (cBKP) hierarchy and the constrained CKP (cCKP) hierarchy play an important role in the commutative integrable systems.

In [7], the Virasoro symmetry and ASvM formula of the BKP hierarchy were given. In [8,9], the gauge transformations of the BKP, CKP, constrained BKP and constrained CKP hierarchies were constructed. In the paper [10], we construct the generalized additional symmetries of the two-component BKP hierarchy and identify its algebraic structure. As a reduction of the two-component BKP hierarchy, the D type Drinfeld-Sokolov hierarchy was found to be a good differential model to derive a complete Block type infinite dimensional Lie algebra (also called Torus Lie algebra). About the Block algebra and its quantization (quantum torus algebra) related to integrable systems, we did a series of works in [11]–[14]. In the paper [15], we constructed the additional symmetries of the supersymmetric BKP hierarchy which constitute a B type $SW_{1+\infty}$ Lie algebra. Further we generalize the SBKP hierarchy to a supersymmetric two-component BKP hierarchy (S2BKP) hierarchy and a new supersymmetric Drinfeld-Sokolov hierarchy of type D which has a super Block type additional symmetry.

There is another kind of generalization of KP and Toda systems called multi-component KP [16,17] or multi-component Toda system which attracts more and more attention because of its widely use in many fields such as the fields of multiple orthogonal polynomials and non-intersecting Brownian motions. In [18], they considered a generalized multicomponent KP hierarchy which contains N independent generalized scalar KP hierarchies in particular by considering a commutative subalgebra of diagonal matrices. In [19], a formalism of multicomponent BKP hierarchies using elementary geometry of spinors was developed by Kac and van de Leur. In [20], M. Mañas, Luis Martínez Alonso construct a relation between multicomponent BKP hierarchy and Lamé equations from Ramond fermions. The τ functions of a $2N$ -multicomponent KP hierarchy provide solutions of the N -multicomponent two dimensional Toda hierarchy [4] which was considered from the point of view of the theory of multiple matrix orthogonal polynomials, non-intersecting Brownian motions and matrix Riemann-Hilbert problem [21]–[22]. The multicomponent Toda hierarchy in [21] is a periodic reduction of bi-infinite matrix-formed Toda hierarchy which contains matrix-formed Toda equation as the first flow equation. In [23], we defined the extended multi-component Toda hierarchy and its Sato theory.

In [24], a new hierarchy called as Z_m -KP hierarchy which take values in a maximal commutative subalgebra of $gl(m, \mathbb{C})$ was constructed, meanwhile the relation between Frobenius manifold and dispersionless reduced Z_m -KP hierarchy was discussed. This inspired us to consider the Hirota quadratic equation of the commutative version of extended multi-component Toda hierarchy in [25] which might be useful in Frobenius manifold theory.

This paper is arranged as follows. In the next section we recall the factorization problem and construct the multicomponent Z_N -BKP hierarchy. In Section 2, we will give the Lax equations of the commutative multicomponent BKP hierarchy. In Section 3, multi-fold transformations of the CMBKP hierarchy will be constructed using the determinant technique in [26,27]. We construct a new constrained CMBKP hierarchy which contains some new integrable systems including a coupled commutative matrix KdV equation in Section 4. In Section 5, the quantum torus symmetry and quantum torus constraint on the tau function of the commutative multi-component BKP hierarchy

will be constructed. Section 6 will be devoted to a short conclusions and discussions.

2 Lax equations of CMBKP hierarchy

In this section we will use the factorization problem to derive Lax equations. We will consider the linear space of the complex $N \times N$ matrix-valued function $g : \mathbb{R} \rightarrow M_N(\mathbb{C})$ with the derivative operator ∂ . Then the set \mathfrak{g} of Laurent series in ∂ as an associative algebra is a Lie algebra under the standard commutator. This Lie algebra has the following important splitting

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \quad (1)$$

where

$$\mathfrak{g}_+ = \left\{ \sum_{j \geq 0} X_j(x) \partial^j, \quad X_j(x) \in M_N(\mathbb{C}) \right\}, \quad \mathfrak{g}_- = \left\{ \sum_{j < 0} X_j(x) \partial^j, \quad X_j(x) \in M_N(\mathbb{C}) \right\}.$$

The splitting (1) leads us to consider the following factorization of $g \in G$

$$g = g_-^{-1} \circ g_+, \quad g_{\pm} \in G_{\pm} \quad (2)$$

where G_{\pm} have \mathfrak{g}_{\pm} as their Lie algebras. G_+ is the set of invertible linear operators of the form $\sum_{j \geq 0} g_j(x) \partial^j$; while G_- is the set of invertible linear operators of the form $1 + \sum_{j < 0} g_j(x) \partial^j$. This algebra has a maximal commutative subalgebra $Z_N = \mathbb{C}[\Gamma]/(\Gamma^N)$ and $\Gamma = (\delta_{i,j+1})_{ij} \in gl(N, \mathbb{C})$. Denote $Z_N(\partial) := \mathfrak{g}_c$, then we have the following splitting

$$\mathfrak{g}_c = \mathfrak{g}_{c+} \oplus \mathfrak{g}_{c-}, \quad (3)$$

where

$$\mathfrak{g}_{c+} = \left\{ \sum_{j \geq 0} X_j(x) \partial^j, \quad X_j(x) \in Z_N \right\}, \quad \mathfrak{g}_{c-} = \left\{ \sum_{j < 0} X_j(x) \partial^j, \quad X_j(x) \in Z_N \right\}.$$

We denote “ $*$ ” as a formal adjoint operation defined by $p^* = \sum (-1)^i \partial^i \circ p_i$ for an arbitrary Z_N -valued pseudo-differential operator $p = \sum p_i \partial^i$, and $(fg)^* = g^* f^*$ for two operators (f, g) . Here \circ means the multiplication of two operators.

Before the work, we list some identities, which will be used in the following sections:

$$A^* = A, \quad (4)$$

$$(AB)^* = BA, \quad (5)$$

$$(A \circ \partial \circ B)^* = -B \circ \partial \circ A, \quad (6)$$

where A and B are $N \times N$ Z_N -valued matrix functions. The Lax operator of the CMBKP hierarchy has form

$$L = \partial + \sum_{i \geq 1} u_i \partial^{-i}, \quad (7)$$

where u_i takes values in the commutative subalgebra Z_N . And the operator must satisfy the following so-called B type condition

$$L^* = -\partial \circ L \circ \partial^{-1}. \quad (8)$$

The CMBKP hierarchy is defined by the following Lax equations:

$$\partial_{2k-1} L = [(B_{2k-1})_+, L], \quad B_{2k-1} = L^{2k-1}, \quad k \geq 1. \quad (9)$$

One can write the operators L in a dressing form as

$$L = \Phi \circ \partial \circ \Phi^{-1}, \quad (10)$$

where

$$\Phi = 1 + \sum_{i \geq 1} a_i \partial^{-i}, \quad (11)$$

satisfy

$$\Phi^* = \partial \circ \Phi^{-1} \circ \partial^{-1}. \quad (12)$$

We call eq.(12) the B type condition of the CMBKP hierarchy. Given L , the dressing operators Φ are determined uniquely up to a multiplication to the right by operators with constant coefficients. The dressing operator Φ takes values in a B type commutative Volterra group in G_- . The CMBKP hierarchy (23) can also be redefined as

$$\frac{\partial \Phi}{\partial t_{2k-1}} = -(L^{2k-1})_- \circ \Phi, \quad (13)$$

with $k \geq 1$. In the the CMBKP hierarchy, we can derive an equation as following

$$9v_{x,t_5} - 5v_{t_3,t_3} + (v_{xxxxx} - 5v_{xx,t_3} - 15v_x v_{t_3} + 15v_x v_{xxx} + 15v_x^3)_x = 0, \quad (14)$$

where $v = \int u_1 dx$ is in the Z_N algebra. We will call the eq.(14) the CMBKP equation. When $N = 2$, we can derive the following two-component CMBKP equation as

$$9w_{x,t_5} - 5w_{t_3,t_3} + (w_{xxxxx} - 5w_{xx,t_3} - 15w_x w_{t_3} + 15w_x w_{xxx} + 15w_x^3)_x = 0, \quad (15)$$

$$9z_{x,t_5} - 5z_{t_3,t_3} + (z_{xxxxx} - 5z_{xx,t_3} - 15w_x z_{t_3} - 15z_x w_{t_3} + 15w_x z_{xxx} + 15z_x w_{xxx} + 45z_x w_x^2)_x = 0, \quad (16)$$

where $v = w + z\Gamma$. After freezing the t_3 flow, the CMBKP equation will be reduced to commutative two-component Sawada-Kotera(CMSK) equation as

$$9w_{x,t_5} + (w_{xxxxx} + 15w_x w_{xxx} + 15w_x^3)_x = 0, \quad (17)$$

$$9z_{x,t_5} + (z_{xxxxx} + 15w_x z_{xxx} + 15z_x w_{xxx} + 45z_x w_x^2)_x = 0. \quad (18)$$

With the above preparation, it is time to construct gauge transformations for the CMBKP hierarchy in the next section.

3 Gauge transformations of the CMBKP hierarchy

In this section, we will consider the gauge transformation of the CMBKP hierarchy on the Lax operator

$$L^{[1]} = \partial + \sum_{i \geq 1} U_i^{[1]} \partial^{-i} = W \circ L \circ W^{-1}, \quad (19)$$

where W is the gauge transformation operator. And $L^{[1]}$ should satisfy the B type condition

$$(L^{[1]})^* = -\partial \circ L^{[1]} \circ \partial^{-1}, \quad (20)$$

which further implies

$$W^* = \partial \circ W^{-1} \circ \partial^{-1}. \quad (21)$$

That means after the gauge transformation, the spectral problem about the $N \times N$ spectral matrix ϕ taking values in the commutative subalgebra Z_N will preserve its form as

$$L \cdot \phi = \lambda \phi, \quad \frac{\partial \phi}{\partial t_n} = B_n \cdot \phi. \quad (22)$$

To keep the Lax pair of the CMBKP hierarchy invariant, i.e.,

$$\partial_{t_n} L^{[1]} = [(B_n^{[1]})_+, L^{[1]}], \quad B_n^{[1]} = (L^{[1]})^n, \quad n = 1, 3, 5, \dots, \quad (23)$$

the dressing operator W should satisfy the following dressing equation

$$W_{t_n} = -W \circ (B_n)_+ + (W \circ B_n \circ W^{-1})_+ \circ W, \quad n = 1, 3, 5, \dots, \quad (24)$$

where W_{t_n} means the derivative of W by t_n .

The evolutions of the eigenfunction ϕ and the adjoint eigenfunction ψ of the CMBKP hierarchy are defined respectively by

$$\frac{\partial \phi}{\partial t_n} = B_n \cdot \phi, \quad \frac{\partial \psi}{\partial t_n} = -(B_n)^* \cdot \psi, \quad (25)$$

where $\phi = \phi(\lambda; t)$ and $\psi = \psi(\lambda; t)$ and $t = (t_1, t_3, t_5, \dots)$. To give the gauge transformation, we need the following lemma.

Lemma 1. *The operator $B := \sum_{n=0}^{\infty} b_n \partial^n$ ($B := \sum_{n=0}^{\infty} \partial^n \circ a_n$) is a Z_N -valued differential operator and f, g (short for $f(x), g(x)$) are two matrix functions taking values in the commutative subalgebra Z_N , following identities hold*

$$(B \circ f \partial^{-1} \circ g)_- = (B \cdot f) \circ \partial^{-1} \circ g, \quad (f \partial^{-1} \circ g \circ B)_- = f \partial^{-1} \circ (B^* \cdot g). \quad (26)$$

Proof. Here we only give the proof of the second equation of (26) by direct calculation basing on the first equation of (26)

$$\begin{aligned}
(f\partial^{-1} \circ g \circ B)_- &= (-B^* \circ g \circ \partial^{-1} \circ f)_-^* \\
&= ((-B^* \cdot g) \circ \partial^{-1} \circ f)^* \\
&= \sum_{m=0}^{\infty} (-a_m((- \partial)^m \cdot g) \partial^{-1} \circ f)^* \\
&= \sum_{m=0}^{\infty} (-1)^m f \partial^{-1} \circ (\partial^m \cdot g)^* \circ a_m \\
&= \sum_{m=0}^{\infty} f \partial^{-1} (-1)^m \circ a_m (\partial^m \cdot g) \\
&= f \partial^{-1} \circ (B^* \cdot g).
\end{aligned} \tag{27}$$

□

Lemma 2. *The operators $T_D = \phi \circ \partial \circ \phi^{-1}$ and $T_I = \psi^{-1} \circ \partial^{-1} \circ \psi$ satisfy eq.(24) , which implies $T_D T_I = \phi \circ \partial \circ \phi^{-1} \circ \psi^{-1} \circ \partial^{-1} \circ \psi$ can also satisfy eq.(24).*

Now, we will find out the gauge transformation operator W of the CMBKP hierarchy. Firstly, we consider the two operators

$$T_D(\phi) = \phi \circ \partial \circ \phi^{-1}, T_I(\psi) = \psi^{-1} \circ \partial^{-1} \circ \psi, \tag{28}$$

where ϕ and ψ are $N \times N$ matrix-valued eigenfunctions taking values in the commutative subalgebra Z_N . Then we have

$$(T_D^{-1}(\phi))^* = -T_I(\phi), (T_I^{-1}(\psi))^* = -T_D(\psi). \tag{29}$$

We can easily get

$$T_D(\phi) \cdot \phi = 0, (T_I^{-1}(\psi))^* \cdot \psi = 0. \tag{30}$$

Similarly to the reference [26], we can consider two sets of matrix functions $\{\phi_i^{(0)}, i = 1, 2, \dots, n; \phi^{(0)}\}$ and $\{\psi_i^{(0)}, i = 1, 2, \dots, n; \psi^{(0)}\}$. For $T_D(\phi) = \phi \circ \partial \circ \phi^{-1}$, we do iterations by the following two steps. For the first step, we consider:

$$T_D^{(1)} = T_D^{(1)}(\phi_1^{(0)}) = \phi_1^{(0)} \circ \partial \circ (\phi_1^{(0)})^{-1}, \tag{31}$$

we define the rule of transformation under $T_D^{(1)}$ as

$$\phi^{(1)} = T_D^{(1)}(\phi_1^{(0)}) \cdot \phi^{(0)}, \psi^{(1)} = (T_D^{(1)}(\phi_1^{(0)}))^*{}^{-1} \cdot \psi^{(0)} = -T_I(\phi_1^{(0)}) \cdot \psi^{(0)}, \tag{32}$$

$$\phi_i^{(1)} = T_D^{(1)}(\phi_1^{(0)}) \cdot \phi_i^{(0)}, \psi_i^{(1)} = (T_D^{(1)}(\phi_1^{(0)}))^*{}^{-1} \cdot \psi_i^{(0)} = -T_I(\phi_1^{(0)}) \cdot \psi_i^{(0)}, \tag{33}$$

where $i \geq 2$ for $\phi_i^{(1)}$ and

$$\psi_i^{(1)} = -T_I(\phi_1^{(0)}) \cdot (\psi_i^{(0)}). \tag{34}$$

For the second step, we consider:

$$T_D^{(2)} = T_D^{(2)}(\phi_2^{(1)}) = \phi_2^{(1)} \circ \partial \circ (\phi_2^{(1)})^{-1}, \quad (35)$$

we define the rule of transformation under $T_D^{(2)}$ as

$$\phi^{(2)} = T_D^{(2)}(\phi_2^{(1)}) \cdot \phi^{(1)}, \psi^{(2)} = (T_D^{(2)}(\phi_2^{(1)}))^{\ast^{-1}} \cdot \psi^{(1)} = -T_I((\phi_2^{(1)})) \cdot \psi^{(1)}, \quad (36)$$

$$\phi_i^{(2)} = T_D^{(2)}(\phi_2^{(1)}) \cdot \phi_i^{(1)}, \psi_i^{(2)} = (T_D^{(2)}(\phi_2^{(1)}))^{\ast^{-1}} \cdot \psi_i^{(1)} = -T_I((\phi_2^{(1)})) \cdot \psi_i^{(1)}, \quad (37)$$

where $i \geq 3$ for $\phi_i^{(2)}$ and

$$\psi_i^{(2)} = -T_I((\phi_2^{(1)})) \cdot (\psi_i^{(1)}). \quad (38)$$

For $T_I(\psi) = \psi^{-1} \circ \partial^{-1} \circ \psi$, it obeys the following iterated rule:

For the first step, we consider:

$$T_I^{(1)} = T_I^{(1)}(\psi_1^{(0)}) = (\psi_1^{(0)})^{-1} \circ \partial^{-1} \circ (\psi_1^{(0)}), \quad (39)$$

$$\phi^{(1)} = T_I^{(1)}(\psi_1^{(0)}) \cdot \phi^{(0)}, \psi^{(1)} = (T_I^{(1)}(\psi_1^{(0)}))^{\ast^{-1}} \cdot \psi^{(0)} = -T_D((\psi_1^{(0)})) \cdot \psi^{(0)}, \quad (40)$$

$$\phi_i^{(1)} = T_I^{(1)}(\psi_1^{(0)}) \cdot \phi_i^{(0)}, \psi_i^{(1)} = (T_I^{(1)}(\psi_1^{(0)}))^{\ast^{-1}} \cdot \psi_i^{(0)} = -T_D((\psi_1^{(0)})) \cdot \psi_i^{(0)}, \quad (41)$$

where $i \geq 2$ for $\psi_i^{(1)}$ and

$$\psi_i^{(1)} = -T_D((\psi_1^{(0)})) \cdot (\psi_i^{(0)}). \quad (42)$$

For the second step, we consider:

$$T_I^{(2)} = T_I^{(2)}(\psi_2^{(1)}) = (\psi_2^{(1)})^{-1} \circ \partial^{-1} \circ (\psi_2^{(1)}), \quad (43)$$

$$\phi^{(2)} = T_I^{(2)}(\psi_2^{(1)}) \cdot \phi^{(1)}, \psi^{(2)} = (T_I^{(2)}(\psi_2^{(1)}))^{\ast^{-1}} \cdot \psi^{(1)} = -T_D(\psi_2^{(1)}) \cdot \psi^{(1)}, \quad (44)$$

$$\phi_i^{(2)} = T_I^{(2)}(\psi_2^{(1)}) \cdot \phi_i^{(1)}, \psi_i^{(2)} = (T_I^{(2)}(\psi_2^{(1)}))^{\ast^{-1}} \cdot \psi_i^{(1)} = -T_D(\psi_2^{(1)}) \cdot \psi_i^{(1)}, \quad (45)$$

where $i \geq 3$ for $\psi_i^{(1)}$ and

$$\psi_i^{(2)} = -T_D(\psi_2^{(1)}) \cdot (\psi_i^{(1)}). \quad (46)$$

It is obvious that a single step of the operator T_D or I_I can not keep the restriction of the B type condition, we use

$$W_1 = T_{1+1} = T_I(\psi_1^{(1)}) \circ T_D(\phi_1^{(0)}), \quad (47)$$

as the gauge transformation operator and we have $L^{[1]} = W_1 L W_1^{-1}$. Let us check whether it satisfies the required constraint

$$(L^{[1]})^* = -\partial L^{[1]} \partial^{-1}, \quad (48)$$

We can calculate

$$\begin{aligned} (L^{[1]})^* &= ((\psi_1^{(1)})^{-1} \circ \partial^{-1} \circ \psi_1^{(1)} \circ \phi_1^{(0)} \circ \partial \circ (\phi_1^{(0)})^{-1} \circ \\ &\quad L \circ \phi_1^{(0)} \circ \partial^{-1} \circ (\phi_1^{(0)})^{-1} \circ (\psi_1^{(1)})^{-1} \circ \partial \circ \psi_1^{(1)})^* \\ &= -(\psi_1^{(1)}) \circ \partial \circ ((\psi_1^{(1)}))^{-1} (\phi_1^{(0)})^{-1} \circ \partial^{-1} \circ (\phi_1^{(0)}) \circ \partial \circ L \\ &\quad \circ \partial^{-1} \circ (\phi_1^{(0)})^{-1} \circ \partial \circ (\phi_1^{(0)}) \circ (\psi_1^{(1)}) \circ \partial^{-1} \circ ((\psi_1^{(1)}))^{-1}, \end{aligned} \quad (49)$$

and

$$\begin{aligned} -\partial L^{[1]} \partial^{-1} &= -\partial \circ ((\psi_1^{(1)}))^{-1} \circ \partial^{-1} \circ \psi_1^{(1)} \circ \phi_1^{(0)} \circ \partial \circ (\phi_1^{(0)})^{-1} \\ &\quad \circ L \circ \phi_1^{(0)} \circ \partial^{-1} \circ (\phi_1^{(0)})^{-1} \circ (\psi_1^{(1)})^{-1} \circ \partial \circ \psi_1^{(1)} \circ \partial^{-1}, \end{aligned} \quad (50)$$

which means in order to keep the constraint $(L^{[1]})^* = -\partial L^{[1]} \partial^{-1}$, T should satisfy the following equation:

$$T_D(\psi_1^{(1)}) T_I(\phi_1^{(0)}) \circ \partial = \partial \circ T_I(\psi_1^{(1)}) T_D(\phi_1^{(0)}), \quad (51)$$

where $\psi_1^{(1)} = -(\phi_1^{(0)})^{-1} \int (\phi_1^{(0)}) \psi_1^{(0)}$ and \int means the integral about spatial variable x .

Then we can acquire the following theorem because the CMBKP hierarchy takes values in a commutative subalgebra just like the case when $N = 1$, i.e. the case of the original BKP hierarchy..

Theorem 1. *The B type condition of the CMBKP hierarchy implies $\psi_1^{(0)}$ and $\phi_1^{(0)}$ have the following relation:*

$$\psi_1^{(0)} = \phi_{1,x}^{(0)}, \quad (52)$$

The B-type reduction of L guarantee that there exists at least one solution $(\phi; \psi)$ which satisfies eq.(52). In fact the above theorem can be generalized to the case of the $gl(N, \mathbb{C})$ -valued multicomponent BKP hierarchy which is not commutative.

The B type condition of the $gl(N, \mathbb{C})$ -valued multicomponent BKP hierarchy implies noncommutative matrices $\psi_1^{(0)}$ and $\phi_1^{(0)}$ have the following relation:

$$((\phi_1^{(0)})^T)^{-2} (\phi_1^{(0)})_x^T \int (\phi_1^{(0)})^T \psi_1^{(0)} - \psi_1^{(0)} - ((\phi_1^{(0)})^T)^{-1} \left(\int (\phi_1^{(0)})^T \psi_1^{(0)} \right) \phi_{1,x}^{(0)} (\phi_1^{(0)})^{-1} + (\phi_1^{(0)})_x^T = 0, \quad (53)$$

where T means the transpose of matrices.

The proof of the eq.(53) will be skipped here because the focus of this paper is about the CMBKP hierarchy. A thorough study on the $gl(N, \mathbb{C})$ -valued multicomponent BKP hierarchy will be contained in another work of ours recently.

Remark: From eq.(52) to eq.(53), one can see clearly the difference of the BKP systems from Z_N to $gl(N, \mathbb{C})$.

In order to keep the B type restriction of the Lax operator of the CMBKP hierarchy, we do iterations of the gauge transformation $W_n = T_{n+n}$. In particular,

$$W_2 = T_{2+2} = T_I(\psi_2^{(3)}) \circ T_D(\phi_2^{(2)}) \circ T_I(\psi_1^{(1)}) \circ T_D(\phi_1^{(0)}), \quad (54)$$

$$W_n = T_{n+n} = T_I(\psi_n^{(2n-1)}) \circ T_D(\phi_n^{(2n-2)}) \circ \dots \circ T_I(\psi_1^{(1)}) \circ T_D(\phi_1^{(0)}), \quad (55)$$

where $\psi_i^{(2n-1)} = -T_I((\phi_n^{(2n-2)})) \cdot (\psi_i^{(2n-2)})$, $\psi_n^{(i)} = (\phi_n^{(i)})_x$. It can be easily checked that $W_n \cdot \phi_i^{(0)}|_{i \leq n} = 0$, $(W_n^{-1})^* \cdot (\psi_i^{(0)})|_{i \leq n} = 0$. The relations $\psi_n^{(i)} = (\phi_n^{(i)})_x$, $n = 1, 2, \dots$ can keep the dressing procedures $W_n = T_{n+n}$, $n = 1, 2, \dots$ always preserving the B type condition of new Lax operators $L^{[n]}$. This is similar as the case of the BKP hierarchy in [8].

We denote $t = (t_1, t_3, t_5, \dots)$ and introduce the Z_N -valued wave function as

$$w(t; z) = \Phi \cdot e^{\xi(t; z)}, \quad (56)$$

where the function ξ is defined as $\xi(t; z) = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} t_k z^k$. It is easy to see

$$L w(t; z) = z w(t; z), \quad \frac{\partial w}{\partial t_{2n+1}} = L_+^{2n+1} w. \quad (57)$$

The Z_N -valued tau function τ of the CMBKP hierarchy can be defined in form of the wave functions as

$$w(t, z) = \frac{\tau(t - 2[z^{-1}])}{\tau(t)} e^{\xi(t; z)}, \quad (58)$$

where $[z] = (z, z^3/3, z^5/5, \dots)$.

The generating functions of n -step T_D and n -step T_I are denoted as $(\phi_1, \dots, \phi_{n-1}, \phi_n)$ and $(\psi_1, \dots, \psi_{n-1}, \psi_n)$ in order respectively. The generating functions have the following B type constraint

$$\psi_i = (\phi_i)_x. \quad (59)$$

Using the above gauge transformation, we can derive the gauge transformation on the tau function of the CMBKP hierarchy as

$$\tau^{(n+n)} = GW_{n,n}(\psi_n, \psi_{n-1}, \dots, \psi_1; \phi_1, \dots, \phi_{n-1}, \phi_n) \tau, \quad (60)$$

where the generalized Wronskian $GW_{k,n}$ is defined as [9]

$$GW_{k,n}(g_k, g_{k-1}, \dots, g_1; f_1, f_2, \dots, f_n) \quad (61)$$

$$= \begin{vmatrix} \int g_k f_1 & \int g_k f_2 & \int g_k f_3 & \dots & \int g_k f_n \\ \int g_{k-1} f_1 & \int g_{k-1} f_2 & \int g_{k-1} f_3 & \dots & \int g_{k-1} f_n \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \int g_1 f_1 & \int g_1 f_2 & \int g_1 f_3 & \dots & \int g_1 f_n \\ f_1 & f_2 & f_3 & \dots & f_n \\ f_{1x} & f_{2x} & f_{3x} & \dots & f_{nx} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ (f_1)^{(n-k-1)} & (f_2)^{(n-k-1)} & (f_3)^{(n-k-1)} & \dots & (f_n)^{(n-k-1)} \end{vmatrix}. \quad (62)$$

When $k = 0$, the generalized Wronskian $GW_{0,n}$ will be reduced to the ordinary Wronskian. Now, we will only give the first gauge transformation of the CMBKP hierarchy included in the following proposition.

Proposition 1. *If the eigenfunction ϕ and the adjoint eigenfunction ψ satisfy the eq.(25), the one-fold gauge transformation operator of the CMBKP hierarchy*

$$W_1 = (\psi_1^{(1)})^{-1} \circ \partial^{-1} \circ \psi_1^{(1)} \circ \phi_1^{(0)} \circ \partial \circ (\phi_1^{(0)})^{-1}, \quad (63)$$

satisfies $W_1\phi_1^{(0)} = 0$ and $(W_1^{-1})^*(\psi_1^{(0)}) = 0$. W_1 will generate new solutions $U_i^{[1]}$ from seed solutions U_i . To see it clearly, here we only give the transformations of first two dynamic functions

$$U_1^{[1]} = U_1 + 2(\ln \phi_1^{(0)})_{xx}, \quad (64)$$

$$U_2^{[1]} = U_2 + 4(\ln \phi_1^{(0)})_{xx}(\ln \phi_1^{(0)})_x - 2\left(\frac{\psi_{1,x}^{(0)}}{\phi_1^{(0)}}\right)_x. \quad (65)$$

If we suppose $U_1 = \begin{bmatrix} \alpha & 0 \\ \beta & \alpha \end{bmatrix}$, $U_2 = \begin{bmatrix} \gamma & 0 \\ \eta & \gamma \end{bmatrix}$, $\phi_1^{(0)} = \begin{bmatrix} \phi_0 & 0 \\ \phi_1 & \phi_0 \end{bmatrix}$, then we can derive the explicit transformation as

$$\alpha^{[1]} = \alpha + 2(\ln \phi_0)_{xx}, \quad (66)$$

$$\beta^{[1]} = \beta + 2\left(\frac{\phi_1}{\phi_0}\right)_{xx}, \quad (67)$$

$$\gamma^{[1]} = \gamma + 4(\ln \phi_0)_{xx}(\ln \phi_0)_x - 2\left(\frac{\phi_{0xx}}{\phi_0}\right)_x, \quad (68)$$

$$\eta^{[1]} = \eta + 4(\ln \phi_0)_{xx}\left(\frac{\phi_1}{\phi_0}\right)_x + 4\left(\frac{\phi_1}{\phi_0}\right)_{xx}(\ln \phi_0)_x - 2\left(\frac{\phi_{1xx}}{\phi_0} - \frac{\phi_{0xx}\phi_1}{\phi_0^2}\right)_x. \quad (69)$$

In the calculation, the identity $\ln \begin{bmatrix} \phi_0 & 0 \\ \phi_1 & \phi_0 \end{bmatrix} = \begin{bmatrix} \ln \phi_0 & 0 \\ \frac{\phi_1}{\phi_0} & \ln \phi_0 \end{bmatrix}$ is used.

For the case $N = 1$, W_1 will generate new solutions of the BKP hierarchy from seed solutions.

4 Constrained CMBKP hierarchy

In this section, we will consider the operator of the constrained CMBKP(cCMBKP) hierarchy as

$$L = \partial + u\partial^{-1}v_x - v\partial^{-1}u_x, \quad (70)$$

where u and v are $N \times N$ matrix functions taking values in Z_N . Here u, v should satisfy the following Sato equation

$$u_{t_{2n-1}} = B_{2n-1} \cdot u, \quad v_{t_{2n-1}} = B_{2n-1} \cdot v. \quad (71)$$

Because the B type condition eq.(8), one can prove that the ∂^0 term does not exist in B_{2n-1} as mentioned in [1]. That means that $u = v = 1$ is a trivial solution.

Suppose $u = q + p\Gamma$, $v = r + s\Gamma$, and consider the case when $N = 2$, i.e.

$$u = \begin{bmatrix} q & 0 \\ p & q \end{bmatrix}, \quad v = \begin{bmatrix} r & 0 \\ s & r \end{bmatrix}. \quad (72)$$

Then we can further derive the following coupled equations

$$\begin{aligned} qt_3 &= q_{3x} + 3(qr_x - rq_x)q_x \\ pt_3 &= p_{3x} + 3[(qr_x - rq_x)p_x + (pr_x + qs_x - sq_x - rp_x)q_x] \\ rt_3 &= r_{3x} + 3(qr_x - rq_x)r_x \\ st_3 &= s_{3x} + 3[(qr_x - rq_x)s_x + (pr_x + qs_x - sq_x - rp_x)r_x]. \end{aligned}$$

If $q = r, p = s$, we can derive the following trivial equations

$$\begin{aligned} q_{t_3} &= q_{3x}, \\ p_{t_3} &= p_{3x}. \end{aligned}$$

If $r = s = 1$, we can derive the following coupled matrix KdV-like equation

$$\begin{aligned} q_{t_3} &= q_{3x} - 3q_x^2, \\ p_{t_3} &= p_{3x} - 6q_x p_x - 3q_x^2. \end{aligned}$$

Similarly as [9], we can derive the new solutions generated from the seed solution q, r

$$u^{(n+n)} = \frac{GW_{n,n+1}(\psi^{(n-1)}, \psi^{(n-2)}, \dots, \psi^{(1)}, u_x; u, \phi^{(1)}, \dots, \phi^{(n-1)}, \phi^{(n)})}{GW_{n,n}(\psi^{(n-1)}, \psi^{(n-2)}, \dots, \psi^{(1)}, u_x; u, \phi^{(1)}, \dots, \phi^{(n-2)}, \phi^{(n-1)})}, \quad (73)$$

$$v^{(n+n)} = \frac{(-1)^n GW_{n-1,n}(\psi^{(n-2)}, \psi^{(n-3)}, \dots, \psi^{(1)}, v_x; v, \phi^{(1)}, \dots, \phi^{(n-2)}, \phi^{(n-1)})}{GW_{n,n}(\psi^{(n-1)}, \psi^{(n-2)}, \dots, \psi^{(1)}, u_x; u, \phi^{(1)}, \dots, \phi^{(n-2)}, \phi^{(n-1)})}, \quad (74)$$

where $\phi^{(j)} = L^j u$ and $(\phi^{(j)}, \psi^{(j)})$ have the same relation as eq.(59). Also the iteration on the constrained tau functions τ_c of the constrained CMBKP hierarchy as

$$\tau_c^{(n+n)} = GW_{n,n}(\psi^{(n-1)}, \psi^{(n-2)}, \dots, \psi^{(1)}, u_x; u, \phi^{(1)}, \dots, \phi^{(n-2)}, \phi^{(n-1)})_{\tau_c}. \quad (75)$$

In the above process of calculations, all the elements in above Wronskians must keep being always written in terms of Γ . In this way, one can keep the new solutions $u^{(n+n)}, v^{(n+n)}$ take values in the algebra Z_N .

5 Quantum torus constraint of CMBKP hierarchy

In this section, we will focus on the quantum torus symmetry of the CMBKP hierarchy. Firstly we define the operator Γ_B and the Z_N -valued Orlov-Shulman's operator M as

$$\Gamma_B = \sum_{i \in \mathbb{Z}_+^{\text{odd}}} it_i \partial^{i-1}, \quad M = \Phi \Gamma_B \Phi^{-1}. \quad (76)$$

The Lax operator L and the Z_N -valued Orlov-Shulman's M operator satisfy the following canonical relation

$$[L, M] = 1. \quad (77)$$

With the above preparation, it is time to construct additional symmetries for the CMBKP hierarchy in the next part. Then it is easy to get that the operator M satisfy

$$[L, M] = 1, \quad Mw(z) = \partial_z w(z); \quad (78)$$

$$\frac{\partial M}{\partial t_k} = [(L^k)_+, M], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (79)$$

Given any pair of integers (m, n) with $m, n \geq 0$, we will introduce the following Z_N -valued operator B_{mn}

$$B_{mn} = M^m L^n - (-1)^n L^{n-1} M^m L. \quad (80)$$

For any Z_N -valued operator B_{mn} in (80), one has

$$\frac{\partial B_{mn}}{\partial t_k} = [(L^k)_+, B_{mn}], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (81)$$

Using

$$\Phi^* = \partial \Phi^{-1} \partial^{-1}, \quad \Gamma_B^* = \Gamma_B; \quad (82)$$

the Z_N -valued operator M satisfies the following identity,

$$M^* = \partial L^{-1} M L \partial^{-1}. \quad (83)$$

It is easy to check that the Z_N -valued operator B_{mn} satisfy the B type condition, namely

$$B_{mn}^* = -\partial B_{mn} \partial^{-1}. \quad (84)$$

Now we will denote the operator D_{mn} as

$$D_{mn} := e^{mM} q^{nL} - L^{-1} q^{-nL} e^{mM} L. \quad (85)$$

Using eq. (84), the B type property of D_{mn} can be derived as

$$D_{mn}^* = -\partial D_{mn} \partial^{-1}.$$

Therefore we get the following important B type condition which the Z_N -valued operator D_{mn} satisfies

$$D_{mn}^* = -\partial D_{mn} \partial^{-1}. \quad (86)$$

Then basing on a quantum parameter q , the additional flows for the time variable $t_{m,n}, t_{m,n}^*$ are defined as follows

$$\frac{\partial \Phi}{\partial t_{m,n}} = -(B_{mn})_- \Phi, \quad \frac{\partial \Phi}{\partial t_{m,n}^*} = -(D_{mn})_- \Phi, \quad (87)$$

or equivalently rewritten as

$$\frac{\partial L}{\partial t_{m,n}} = -[(B_{mn})_-, L], \quad \frac{\partial M}{\partial t_{m,n}^*} = -[(D_{mn})_-, M]. \quad (88)$$

Generally, one can also derive

$$\partial_{t_{l,k}^*} (D_{mn}) = [-(D_{lk})_-, D_{mn}]. \quad (89)$$

Using the similar proof as the BKP hierarchy in [14], the additional flows of $\partial_{t_{l,k}^*}$ can be proved to be symmetries of the CMBKP hierarchy, i.e. they commute with all ∂_{t_n} flows of the CMBKP hierarchy.

The additional flows $\partial_{t_{l,k}^*}$ of the CMBKP hierarchy form the W_∞ algebra similarly as [7] which is about the BKP hierarchy.

Now it is time to identity the algebraic structure of the additional $t_{l,k}^*$ flows of the CMBKP hierarchy.

Theorem 2. *The additional flows $\partial_{t_{l,k}^*}$ of the CMBKP hierarchy form the positive half of quantum torus algebra, i.e.,*

$$[\partial_{t_{n,m}^*}, \partial_{t_{l,k}^*}] = (q^{ml} - q^{nk}) \partial_{t_{n+l, m+k}^*}, \quad n, m, l, k \geq 0. \quad (90)$$

Remark: The $t_{l,k}^*$ additional flows constitute a nice quantum torus algebra because its basing on a commutative algebra. This is different from the multicomponent BKP whose additional symmetry constitute multi-fold quantum torus algebra [28].

Next, similar to the KP and BKP hierarchy [14], we will consider the quantum torus constraint on the Z_N -valued tau function of the CMBKP hierarchy.

Similar as [14], one has shown that

$$\partial_{t_{p,s}} \log w = (e^{\tilde{\eta}} - 1) \frac{Z_s^{(p+1)}(\tau)}{\tau}, \quad (91)$$

where

$$\tilde{\eta} = \sum_{i \in \mathbb{Z}_+^{\text{odd}}} \frac{\lambda^{-i}}{i} \frac{\partial}{\partial t_i}, \quad (92)$$

and $Z_s^{(p+1)}$ is the generator of the W_∞^B algebra. Then with the help of rewriting the quantum torus flow $\partial_{t_{l,k}^*}$ in terms of the $\partial_{t_{p,s}}$ flows

$$\partial_{t_{l,k}^*} = \sum_{p,s=0}^{\infty} \frac{l^p (k \log q)^s}{p! s!} \partial_{t_{p,s}},$$

and denoting

$$L_{l,k}^B := \sum_{p,s=0}^{\infty} \frac{l^p (k \log q)^s}{p! s!} \frac{Z_s^{(p+1)}}{p+1}, \quad (93)$$

the quantum torus constraint on the Z_N -valued wave function w , i.e.

$$\partial_{t_{l,k}^*} w = 0, \quad (94)$$

will lead to the quantum torus constraint on the Z_N -valued tau function of the CMBKP hierarchy

$$L_{l,k}^B \tau = c, \quad (95)$$

where c is a constant.

6 Conclusions and Discussions

In this paper, we defined a new multi-component BKP hierarchy which takes values in a commutative subalgebra of $gl(N, \mathbb{C})$. After this, we give the gauge transformation of the commutative multi-component BKP hierarchy. Meanwhile we construct a new constrained CMBKP hierarchy which contains some integrable systems including coupled matrix KdV equations under a certain reduction. After this, the quantum torus symmetry

and quantum torus constraint of the commutative multi-component BKP hierarchy are constructed. We are looking forward to the possible application of the quantum torus constraint in the topological field theory and enumerate geometry. For the importance of the BKP hierarchy in representation theory and mathematical physics, what is the application of the commutative multi-component BKP hierarchy in other theories such as Frobenius manifold is an interesting question.

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